OPTIMIZATION OF TWO-DIMENSIONAL NONSTEADY-STATE TEMPERATURE REGIMES WITH LIMITATION IMPOSED ON THE PARAMETERS OF THE THERMAL PROCESS

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The two-dimensional nonsteady-state problem of optimum high-speed control of the heating of solids, where limitations are imposed on the control of the body temperature, the temperature drop, and similar parameters is studied.

Problems of optimizing the high-speed heating of solids, as described by the one-dimensional nonsteady-state equation of thermal conductivity, where various limitations are imposed on the phase coordinates, were examined in [1-5]. In [6] we find the conditions adequate for optimum high-speed n-dimensional diffusion-type processes, in particular, thermal conductivity. Based on the results of this reference, we propose in the following a method for the construction of an optimum control mechanism for the heating of solids, described by a two-dimensional nonsteady-state equation of thermal conductivity.

The temperature field $\Theta(x, y, t)$ in these solids satisfies the following boundary-value problem:

$$\Delta\Theta = \frac{1}{a}\frac{\partial\Theta}{\partial t}\ ((x, y, t) \in D = V \times]0, T]); \tag{1}$$

$$\Theta(x, y, 0) = f(x, y)\ ((x, y) \in \overline{V}); \tag{2}$$

$$\lambda\frac{\partial\Theta}{\partial\nu} = \alpha[\Theta - q]\ ((x, y, t) \in S = \partial V \times]0, T]). \tag{3}$$

Here $q(x, y, t)$ is the control function (the temperature of the heating medium), bounded from above:

$$q(x, y, t) \leqslant u(x, y, t)\ ((x, y, t) \in S). \tag{4}$$

Moreover, we must bear in mind during the heating the limitation imposed on the parameters of the thermal process at the surface of the material, and in general from these can be given by the inequality

$$F\Theta \leqslant l(x, y, t)\ ((x, y, t) \in S), \tag{5}$$

where F is the operator which determines these parameters, while $\ell \in C(S)$ denotes their given maximum permissible values. Usually, these parameters include:

a) the temperature of the material

$$F\Theta = \Theta(x, y, t);$$

b) the maximum temperature difference

$$F\Theta = \Theta(x, y, t) - \min_{(x,y)\in\overline{V}} \Theta(x, y, t);$$

c) the flow of heat to the surface of the material

$$F\Theta = \partial\Theta/\partial\nu.$$

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We are now confronted with the following optimization problem. It becomes necessary to determine such a control function $q = \omega$, which within the minimum time $\tau_0 \in]0, T]$, under limitation (4) and (5), will change the material of the solid from the initial state (2) to the final state with the mean-integral temperature Θ_s:

$$\frac{1}{V}\int_V \Theta(x, y, \tau_0)\,dV = \Theta_s \tag{6}$$

or with a given temperature Θ_* at some point $(x_*, y_*) \in \bar{V}$:

$$\Theta(x_*, y_*, \tau_0) = \Theta_*. \tag{7}$$

According to the results of [6], in order for the function $\omega(x, y, t)$ to be optimum with respect to high-speed control, it is enough to satisfy the inequality

$$(F\Theta_\omega - l)(\omega - u) = 0\,((x, y, t) \in S). \tag{8}$$

Here Θ_ω is the solution of the boundary-value problem (1)-(3) when $q = \omega$.

Equality (8) indicates the sought optimum control $\omega(x, y, t)$ at each point on the surface of the material when $t \in]0, \tau_0]$ is either equal to the maximum possible value of the limited parameters, or it allows for the maximum permissible values of these, and the control can be written in the form of two equalities:

$$\omega(x, y, t) = u(x, y, t)\,((x, y, t) \in S_u), \tag{9}$$

$$F\Theta_\omega = l(x, y, t)\,((x, y, t) \in S_l), \tag{10}$$

where $S_u \cup S_\ell = S$.

Then, for purposes of determining the the temperature regime optimum from the standpoint of rapid action, it is enough to find the solution for Eq. (1) which satisfies initial condition (2), as well as condition (3) when we have $q = \omega$ at the surface S_u, as well as satisfaction of condition (10) at the surface S_ℓ, while the optimum control at the surface S_ℓ is determined by substitution of the solution Θ_ω into Eq. (3).

Since the surfaces S_ℓ and S_u have not been determined in advance and their form depends on the initial parameters and functions, as well as on the temperature field, the problem under consideration is essentially nonlinear.

We use the finite-difference method to solve the formulated problem. With this purpose in mind, the region D is covered by a grid with nodes $\{(x_n, y_m, t_k),\ 0 \le n \le N,\ 0 \le m \le M,\ 0 \le k \le K\}$, where in dependence on the geometry of the region V the intervals $h_{1n} = x_{n+1} - x_n$ and $h_{2m} = y_{m+1} - y_m$ with respect to the variables x and y need not be taken to be constant. In the latter case, the grid will be nonuniform.

As is well known [7], for the solution of two-dimensional nonsteady-state equations of thermal conductivity it is expedient to use the so-called economical schemes, since they are absolutely stable and require a comparatively limited number of calculations in the transition from one time level to another. Therefore, for purposes of approximating boundary-value problem (1)-(3), we choose a scheme of variable directions, which contains values of the functions both for the entire layer, and namely: $t_k = k\tau$, as well as for half of the entire layer: $t_{k+1/2} = t_k + \tau/2$ $(k = 0, \ldots, K)$, where τ denotes the interval for the variable $t \in]0, T]$.

The finite-difference analog of boundary-value problem (1)-(3), (9), (10) in accordance with this scheme is written in the form:

$$\frac{1}{a0{,}5\tau}\left(\Theta_{nm}^{(k+1/2)} - \Theta_{nm}^{(k)}\right) = L_1\Theta_{nm}^{(k+1/2)} + L_2\Theta_{nm}^{(k)}, \tag{11}$$

$$\frac{1}{a0{,}5\tau}\left(\Theta_{nm}^{(k+1)} - \Theta_{nm}^{(k+1/2)}\right) = L_1\Theta_{nm}^{(k+1/2)} + L_2\Theta_{nm}^{(k+1)};$$

$$\Theta_{nm}^{(0)} = f_{nm}\,((x_n, y_m) \in \bar{V}); \tag{12}$$

$$\Phi_{nm}\Theta = \frac{\alpha}{\lambda}\left[\Theta_{nm}^{(k)} - \omega_{nm}^{(k)}\right]((x_n, y_m, t_k) \in P^{(k)} \subset Z); \tag{13}$$

$$F_{nm}\Theta = l_{nm}^{(k)}((x_n,\ y_m,\ t_k)\in R^{(k)} \subset Z). \tag{14}$$

Here

$$L_1\Theta_{nm}^{(k)} = a_n\Theta_{n+1,m}^{(k)} - c_n\Theta_{nm}^{(k)} + b_n\Theta_{n-1,m}^{(k)};$$

$$L_2\Theta_{nm}^{(k)} = \alpha_m\Theta_{n,m+1}^{(k)} - \gamma_m\Theta_{nm}^{(k)} + \beta_m\Theta_{n,m-1}^{(k)};$$

a_n, b_n, c_n, α_m, β_m, γ_m ($n = 1, \ldots, N - 1$, $m = 1, \ldots, M - 1$) are the variable coefficients whose values for the case of polar and rectangular coordinates are given in [8]; $\Theta_{nm}^{(k)}$, f_{nm}, $\omega_{nm}^{(k)}$, and $\ell_{nm}^{(k)}$ are the values of the functions Θ, f, ω, and ℓ at the nodes of the grids (x_n, y_m, t_k); Z is the set of points of the finite-difference approximation of the boundary ∂V of region V; $P^{(k)}$ and $R^{(k)}$ denote the sets of nodes of the approximations making up the region $S_u \cap \{t = t_k\}$ and $S_\ell \cap \{t = t_k\}$, so that consequently, $P^{(k)} \cup R^{(k)} = Z$; F_{nm}, Φ_{nm} are the corresponding finite-difference analogs of the operators F and $\partial/\partial\nu$.

The idea behind the method of variable directions lies in the fact that if the values of $\Theta_{nm}^{(k)}$ ($n = 0, \ldots, N$; $m = 0, \ldots, M$) are known, then initially from the first of the equations in (11) and boundary conditions (13) and (14) at the semicomplete layer we determine the values of $\Theta_{nm}^{(k+1/2)}$, and then from the second of the equations in (11) and the same boundary conditions, we determine the values of $\Theta_{nm}^{(k+1)}$, and for their determination it is enough to make use of the sweeping method [9].

The difficulty involved in using this method to solve the stated problem involves the fact that, as was mentioned earlier, the boundaries of the regions S_ℓ and S_u are not known in advance and, consequently, we do not know the sets $R^{(k+1)}$ and $P^{(k+1)}$, needed to calculate the values of $\Theta_{nm}^{(k+1)}$.

To eliminate this difficulty, we propose an iteration procedure which allows us to construct a sequence of sets $P_i^{(k+1)}$ and $R_i^{(k+1)}$ ($1 \le i < \infty$), which within a finite number of iterations converges to the sought sets $P^{(k+1)}$ and $R^{(k+1)}$.

At some k-th time layer, let the values of $\Theta_{nm}^{(k)}$ be known and, consequently, the sets $P^{(k)}$ and $R^{(k)}$. We will choose $P_1^{(k+1)} = P^{(k)}$, $R_1^{(k+1)} = R^{(k)}$. Then, if as a result of these calculations the derived values of $\Theta_{nm}^{(k+1)}$ on the set $P_1^{(k+1)}$ satisfy condition (5), and on the set $R_1^{(k+1)}$ they satisfy condition (4), it is assumed that $P^{(k+1)} = P_1^{(k+1)}$, $R^{(k+1)} = R_1^{(k+1)}$ and we make the transition to calculation of the temperature values on the next time level. In the opposite case, we form the set $P_2^{(k+1)}$ in accordance with the following principle: it contains all of the points from $P_1^{(k+1)}$, at which condition (5) is satisfied, as well as those points from $R_1^{(k+1)}$, at which condition (4) is not satisfied. We then determine the set $R_2^{(k+1)} = Z\backslash P_2^{(k+1)}$ and we repeat the calculation of the values of $\Theta_{nm}^{(k+1)}$.

This procedure is repeated until we obtain the sets $P^{(k+1)} = P_i^{(k+1)}$ and $R^{(k+1)} = R_i^{(k+1)}$, on each of which conditions (5) and (4) are completely satisfied, respectively.

Since we know the values of $\Theta_{nm}^{(0)} = f(x_n, y_m)$ from (12), and for a solution to exist for the optimization problem (1)-(7) we must satisfy the condition

$$Ff \leqslant l(x,\ y,\ 0)\ ((x,\ y)\in \bar{V}),$$

so that we have $P^{(0)} = Z$, $R^{(0)} = \varnothing$, and, using the above-described procedure, we can determine the values of $\Theta_{nm}^{(k)}$ in the case of $k = 1, \ldots, K$. The values of the optimum control $\omega_{nm}^{(k)}$ on the set $R^{(k)}$ are determined from the formula

$$\omega_{nm}^{(k)} = \Theta_{nm}^{(k)} - \frac{\lambda}{\alpha}\Phi_{nm}\Theta.$$

Thus, on the basis of the variable-directions method, we have constructed an algorithm which makes it possible to find a solution for the optimization problem (1)-(7).

We examined the optimization problem for high-speed heating of a hollow cylinder as an example, with limitations imposed on the temperature of the heated surface.

It was assumed that the temperature field of the cylinder satisfies the following boundary-value problem:

$$\frac{\partial^2\Theta}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\Theta}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\Theta}{\partial\varphi^2} = \frac{\partial\Theta}{\partial\tau}$$

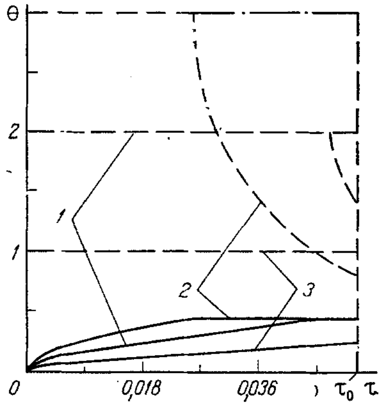

Fig. 1

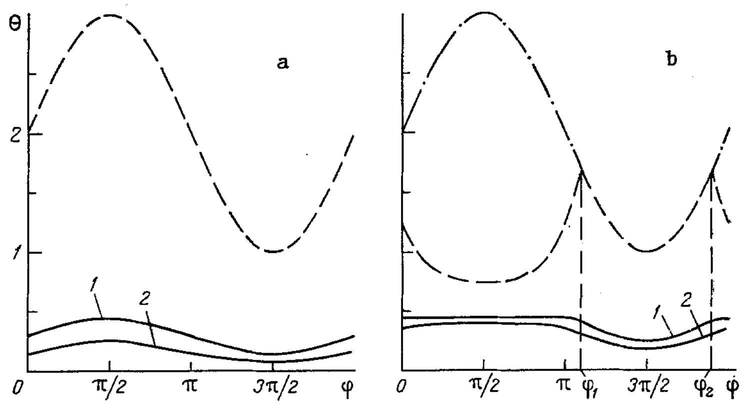

Fig. 2

Fig. 1. Change over time in the values of temperature and the control functions: 1) $\rho = 1$, $\varphi = 0$; 2) $\rho = 1$, $\varphi = \pi/2$; 3) $\rho = 1$, $\varphi = 3\pi/2$.

Fig. 2. Distribution of temperature values and of the control functions in terms of the angular coordinate φ at the instant at which the maximum permissible temperature of the body ℓ_* is attained: a) $\tau = 2.55 \cdot 10^{-2}$, and the ultimate heating: b) $\tau = 5.13 \cdot 10^{-2}$. 1) $\rho = 1$; 2) $\rho = k$.

$$((\rho, \varphi, \tau) \in [k, 1] \times [0, 2\pi] \times]0, T]);$$

$$\Theta(\rho, \varphi, 0) \equiv 0;$$

$$\left.\frac{\partial \Theta}{\partial \rho}\right|_{\rho=1} = -H_* [\Theta(1, \varphi, \tau) - q(\varphi, \tau)];$$

$$\left.\frac{\partial \Theta}{\partial \rho}\right|_{\rho=k} = 0,$$

and the control $q(\varphi, \tau)$ is bounded from above by the function $2 + \sin\varphi$.

The temperature at the outside surface of the cylinder ($\rho = 1$) should not exceed ℓ_* during the heating process, and it is our ultimate goal to achieve the minimum temperature at this same surface such that the temperature difference does not exceed the value of ε, i.e.,

$$\min_{\varphi\in[0,2\pi]} \Theta(1, \varphi, \tau_0) = l_* - \varepsilon.$$

The following values were assumed for the parameters in our calculations: $k = 0.8$, $H_* = 0.8$, $\ell_* = 0.44$, $\varepsilon = 0.2$.

Figures 1 and 2 show the results from the calculation of the optimum control (dashed lines) and the corresponding temperature values (solid lines). The dash–dot lines show the maximum possible values of the control function.

As we can see from Fig. 1, the control with values of $\varphi = 0$ and $\varphi = \pi/2$ is two-staged. In the first stage, it is equal to the maximum possible value, while in the second stage it achieves equivalence with the given temperature value. At the point $(1, 3\pi/2)$ the control is equal to the maximum possible value throughout the entire duration of the heating process, since the value of the temperature at the given point does not exceed ℓ_*.

Figure 2b shows that the optimum control for $\varphi\in[\varphi_1, \varphi_2]$ is equal to the maximum possible value, while in the case of $\varphi\in[0, \varphi_1[\cup]\varphi_2, 2\pi]$ it is lower than this value and results in the maintenance of the given temperature ℓ_*.

It should be noted that although the method covered in this article for the optimization of the heating process referred to uniform materials, it can easily be generalized to the case of piecewise-uniform or nonuniform materials.

NOTATION

x, y, t, spatial coordinates, m, and time, sec, respectively; $\Theta(x, y, t)$, temperature of the material, K; $\bar{V} = \partial V \cup V$; V, region of change in the variables x and y; ∂V, boundary of the region V; ν, internal normal to the surface ∂V; a, λ, and α, respectively, the coefficients of thermal diffusivity, m^2/sec, thermal conductivity, $W/(m\cdot K)$, and heat exchange, $W/(m^2\cdot K)$; $\omega(x, y, t)$, the sought control function (the temperature of the heating medium), K; $\rho = x/R_2$, φ, and $\tau = at/R_2^2$, dimensionless coordinates and time; $k = R_1/R_2$, $H_* = \alpha R_2/\lambda$, R_1 and R_2, the inside and outside radii of the hollow cylinder, m.

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